

## K3 SURFACES WITH AN ORDER 60 AUTOMORPHISM

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ABSTRACT. In characteristic  $p = 0$  or  $p > 5$ , we show that a K3 surface with an order 60 automorphism is unique up to isomorphism. As a consequence, the Fermat quartic surface in characteristic  $p = 11 \bmod 12$  is characterized by a cyclic symmetry of order 60.

Let  $X$  be a K3 surface over an algebraically closed field  $k$  of characteristic  $p \geq 0$ . An automorphism  $g$  of  $X$  is called *symplectic* if it preserves a regular 2-form  $\omega_X$ , and called *purely non-symplectic* if no power of  $g$  is symplectic.

Over  $k = \mathbb{C}$ , based on Nikulin's work [9] Machida and Oguiso [8] proved that a positive integer  $N$  is the order of a purely non-symplectic automorphism of a complex K3 surface if and only if  $\phi(N) \leq 20$  and  $N \neq 60$ , where  $\phi$  is the Euler function. On the other hand, there is a K3 surface with an automorphism of order 60 [7]:

$$(0.1) \quad X_{60} : y^2 + x^3 + t_0 t_1^{11} - t_0^{11} t_1 = 0,$$

$$(0.2) \quad g_{60}(t_0, t_1, x, y) = (t_0, \zeta_{10} t_1, \zeta_{30} x, \zeta_{20} y)$$

where  $\zeta_a \in k$  is a primitive  $a$ -th root of unity. The K3 surface  $X_{60}$  is defined over the integers and both the surface and the automorphism have a good reduction mod  $p$  unless  $p = 2, 3, 5$ .

For an automorphism  $g$  of finite order of a K3 surface  $X$ , we write

$$\text{ord}(g) = m.n$$

if  $g$  is of order  $mn$  and the natural homomorphism  $\langle g \rangle \rightarrow \text{GL}(H^0(X, \Omega_X^2))$  has kernel of order  $m$  and image of order  $n$ .

The main result of the paper is the following.

**Theorem 0.1.** *Let  $k$  be an algebraically closed field of characteristic  $p = 0$  or  $p > 5$ . Let  $X$  be a K3 surface defined over  $k$  with an automorphism  $g$  of order 60. Then*

- (1)  $\text{ord}(g) = 5.12$ ;
- (2) *the pair  $(X, g)$  is isomorphic to the pair  $(X_{60}, g_{60})$ .*

In any given characteristic  $p \geq 0$ , the author [7] has shown that a positive integer  $N$  is the order of a tame automorphism of a K3 surface if and only if

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$\phi(N) \leq 20$ , where  $\phi$  is the Euler function. Here an automorphism of finite order is called *tame* if the characteristic  $p = 0$  or if its order is prime to the characteristic  $p > 0$ . A natural question arises:

**Question:** *In any given characteristic  $p \geq 0$ , which pair  $(m, n)$  realizes as  $\text{ord}(g) = m.n$  for some tame automorphism  $g$  of a K3 surface?*

The answer is given in Section 3 [ibid] for many orders  $N$ . E.g., if  $N$  satisfies  $\phi(N) \geq 16$  and  $N \neq 60, 40$ , then every tame automorphism  $g$  of order  $N$  of a K3 surface is purely non-symplectic, i.e.,  $\text{ord}(g) = 1.N$ . If  $N = 60$ , then Theorem 0.1 shows that the pair  $(m, n) = (5, 12)$  in any characteristic  $p \neq 2, 3, 5$ . The proof of [8] for the nonexistence of a purely non-symplectic automorphism of order 60 of a complex K3 surface depends on the holomorphic Lefschetz formula and the integral lattice theory, hence does not extend to the positive characteristic.

It is well known that the Fermat quartic surface

$$x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0$$

is a supersingular K3 surface with Artin invariant 1, if the characteristic  $p = 3 \bmod 4$ . This can be seen by using the algorithm for determining the Artin invariant of a weighted Delsarte surface whose minimal resolution is a K3 surface ([12], [4]). Using the same algorithm we see that in characteristic  $p = 11 \bmod 12$  the surface  $X_{60}$  is a supersingular K3 surface with Artin invariant 1, hence is isomorphic to the Fermat quartic surface.

**Corollary 0.2.** *In characteristic  $p = 11 \bmod 12$ , the Fermat quartic surface is the only K3 surface with an order 60 automorphism.*

Over  $k = \mathbb{C}$ , Oguiso [10] proved that the Fermat quartic surface is the only K3 surface with a faithful action of a nilpotent group of order  $512 = 2^9$ . Over  $k = \mathbb{C}$ , the surface  $X_{60}$  is not isomorphic to the Fermat quartic surface, as the former admits a purely non-symplectic automorphism of order 12, while the latter has Picard number 20 and transcendental rank 2, hence by Nikulin [9] does not admit a purely non-symplectic automorphism of order  $n$  with  $\phi(n) > 2$ .

**Remark 0.3.** In characteristic  $p = 11$  the surface  $X_{60}$  or the Fermat quartic surface also admits a cyclic action of order 66 [7] and a symplectic action of the simple groups  $M_{22}$ ,  $M_{11}$  and  $L_2(11)$  [3], where  $M_r$  is one of the Mathieu groups.

## Notation

For an automorphism  $g$  of a K3 surface  $X$ , we use the following notation:

- $\text{NS}(X)$  : the Néron-Severi group of  $X$
- $X^g = \text{Fix}(g)$  : the fixed locus of  $g$
- $[g^*] = [\lambda_1, \dots, \lambda_{22}]$  : the eigenvalues of  $g^*|H_{\text{et}}^2(X, \mathbb{Q}_l)$

- $\zeta_a$  : a primitive  $a$ -th root of unity in  $\overline{\mathbb{Q}_l}$
- $[\zeta_a : \phi(a)] \subset [g^*]$  : all primitive  $a$ -th roots of unity appear in  $[g^*]$  where  $\phi(a)$  indicates the number of them.
- $[\lambda.r] \subset [g^*]$  :  $\lambda$  repeats  $r$  times in  $[g^*]$ .
- $[(\zeta_a : \phi(a)).r] \subset [g^*]$  : the list  $\zeta_a : \phi(a)$  repeats  $r$  times in  $[g^*]$ .

## 1. PRELIMINARIES

Let  $X$  be a K3 surface over an algebraically closed field  $k$  of positive characteristic  $p$ . We first recall the following basic result.

**Proposition 1.1.** (3.7.3 [6], cf. [7]) *Let  $X$  be a projective variety over an algebraically closed field  $k$  of characteristic  $p > 0$ . Let  $g$  be an automorphism of  $X$ . Let  $l \neq p$ . Then the following hold true.*

- (1) *The characteristic polynomial of  $g^*|H_{\text{et}}^j(X, \mathbb{Q}_l)$  has integer coefficients for each  $j$ . In particular,*
  - (a) *the trace of  $g^*|H_{\text{et}}^j(X, \mathbb{Q}_l)$  is an integer;*
  - (b) *if  $\lambda$  is an eigenvalue for  $g^*|H_{\text{et}}^j(X, \mathbb{Q}_l)$ , then the minimal polynomial of  $\lambda$  over  $\mathbb{Z}$  divides the characteristic polynomial.*
- (2) *If  $g$  is of finite order, then  $g$  has an invariant ample divisor, corresponding to a non-zero  $g^*$ -invariant vector in  $H_{\text{et}}^2(X, \mathbb{Q}_l)$ . In particular, 1 is an eigenvalue of  $g^*|H_{\text{et}}^2(X, \mathbb{Q}_l)$ .*
- (3) *If  $g^*$  acts trivially on  $H_{\text{et}}^2(X, \mathbb{Q}_l)$ , then it acts trivially on the space of regular 2-forms  $H^0(X, \Omega_X^2)$ .*
- (4) *If  $g$  is tame and the action of  $g^*$  on  $H^0(X, \Omega_X^2)$  has  $\zeta_n \in k$  as an eigenvalue, then the action of  $g^*$  on  $H_{\text{et}}^2(X, \mathbb{Q}_l)$  has  $\zeta_n \in \overline{\mathbb{Q}_l}$  as an eigenvalue.*

Recall that for a nonsingular projective variety  $Z$  in characteristic  $p > 0$ , there is an exact sequence of  $\mathbb{Q}_l$ -vector spaces

$$(1.1) \quad 0 \rightarrow \text{NS}(Z) \otimes \mathbb{Q}_l \rightarrow H_{\text{et}}^2(Z, \mathbb{Q}_l) \rightarrow T_l^2(Z) \rightarrow 0$$

where  $T_l^2(Z) = T_l(\text{Br}(Z))$  in the standard notation in the theory of étale cohomology. The Brauer group  $\text{Br}(Z)$  is known to be a birational invariant, and it is trivial when  $Z$  is a rational variety. In fact, one can show that

$$\begin{aligned} \text{NS}(Z) \otimes \mathbb{Q}_l &= \text{Ker}(H_{\text{et}}^2(Z, \mathbb{Q}_l) \rightarrow H^2(k(Z), \mathbb{Q}_l)); \\ T_l^2(Z) &= \text{Im}(H_{\text{et}}^2(Z, \mathbb{Q}_l) \rightarrow H^2(k(Z), \mathbb{Q}_l)). \end{aligned}$$

Here  $H^2(k(Z), \mathbb{Q}_l) = \varinjlim_U H^2(U, \mathbb{Q}_l)$ , where  $U$  runs through the set of open subsets of  $Z$  (see [11]). It is known that the dimension of all  $\mathbb{Q}_l$ -spaces from above do not depend on  $l$  prime to the characteristic  $p$ .

**Proposition 1.2.** [7] *In the situation as above, let  $g$  be an automorphism of  $Z$  of finite order. Assume  $l \neq p$ . Then the following assertions are true.*

- (1) *Both traces of  $g^*$  on  $\text{NS}(Z)$  and on  $T_l^2(Z)$  are integers.*
- (2)  *$\text{rank NS}(Z)^g = \text{rank NS}(Z/\langle g \rangle)$ .*

- (3)  $\dim H_{\text{et}}^2(Z, \mathbb{Q}_l)^g = \text{rank NS}(Z)^g + \dim T_l^2(Z)^g$ .  
 (4) If the minimal resolution  $Y$  of  $Z/\langle g \rangle$  has  $T_l^2(Y) = 0$ , then  

$$\dim H_{\text{et}}^2(Z, \mathbb{Q}_l)^g = \text{rank NS}(Z)^g.$$

The condition of (4) is satisfied if  $Z/\langle g \rangle$  is rational or is birational to an Enriques surface.

The following is well known, see e.g., Theorem 3.2 [1].

**Proposition 1.3.** (*Lefschetz fixed point formula*) Let  $X$  be a smooth projective variety over an algebraically closed field  $k$  of characteristic  $p > 0$  and let  $g$  be a tame automorphism of  $X$ . Then  $X^g = \text{Fix}(g)$  is smooth and

$$e(X^g) = \sum_j (-1)^j \text{Tr}(g^* | H_{\text{et}}^j(X, \mathbb{Q}_l)).$$

A tame symplectic automorphism  $h$  of a K3 surface has finitely many fixed points, the number of fixed points  $f(h)$  depends only on the order of  $h$  and the list of possible pairs  $(\text{ord}(h), f(h))$  is the same as in the complex case (Theorem 3.3 and Proposition 4.1 [2]):

$$(\text{ord}(h), f(h)) = (2, 8), (3, 6), (4, 4), (5, 4), (6, 2), (7, 3), (8, 2).$$

Thus by Proposition 1.3, we obtain the following.

**Lemma 1.4.** [7] Let  $h$  be a tame symplectic automorphism of a K3 surface  $X$ . Then  $h^* | H_{\text{et}}^2(X, \mathbb{Q}_l)$  has eigenvalues

$$\begin{aligned} \text{ord}(h) = 2 & : [h^*] = [1, 1.13, -1.8] \\ \text{ord}(h) = 3 & : [h^*] = [1, 1.9, (\zeta_3 : 2).6] \\ \text{ord}(h) = 4 & : [h^*] = [1, 1.7, (\zeta_4 : 2).4, -1.6] \\ \text{ord}(h) = 5 & : [h^*] = [1, 1.5, (\zeta_5 : 4).4] \\ \text{ord}(h) = 6 & : [h^*] = [1, 1.5, (\zeta_3 : 2).4, (\zeta_6 : 2).2, -1.4] \\ \text{ord}(h) = 7 & : [h^*] = [1, 1.3, (\zeta_7 : 6).3] \\ \text{ord}(h) = 8 & : [h^*] = [1, 1.3, (\zeta_8 : 4).2, (\zeta_4 : 2).3, -1.4] \end{aligned}$$

where the first eigenvalue corresponds to an invariant ample divisor.

**Lemma 1.5.** [7] Let  $X$  be a K3 surface in characteristic  $p \neq 2, 3$ . Assume that  $h$  is an automorphism of order 2 with  $\dim H_{\text{et}}^2(X, \mathbb{Q}_l)^h = 2$ . Then  $h$  is non-symplectic and has a  $h$ -invariant elliptic fibration  $\psi : X \rightarrow \mathbb{P}^1$  with 12 cuspidal fibers, and  $X^h$  consists of either a curve of genus 9 which is a 4-section of  $\psi$  passing through each cusp with multiplicity 3 or a section and a curve of genus 10 which is a 3-section passing through each cusp with multiplicity 3. In the second case,  $X/\langle h \rangle$  is the rational ruled surface  $\mathbb{F}_4$ .

We will use frequently the Weyl theorem of the following form.

**Lemma 1.6.** [7] Let  $V$  be a finite dimensional vector space over a field of characteristic 0. Let  $g \in GL(V)$  be a linear automorphism of finite order. Assume that the characteristic polynomial of  $g$  has integer coefficients. If for some positive integer  $m$  a primitive  $m$ -th root of unity appears with multiplicity  $r$  as an eigenvalue of  $g$ , then so does each of its conjugates.

The following easy lemma also will be used frequently.

**Lemma 1.7.** [7] *Let  $S$  be a set and  $\text{Aut}(S)$  be the group of bijections of  $S$ . For any  $g \in \text{Aut}(S)$  and positive integers  $a$  and  $b$ ,*

- (1)  $\text{Fix}(g) \subset \text{Fix}(g^a)$ ;
- (2)  $\text{Fix}(g^a) \cap \text{Fix}(g^b) = \text{Fix}(g^d)$  where  $d = \gcd(a, b)$ ;
- (3)  $\text{Fix}(g) = \text{Fix}(g^a)$  if  $\text{ord}(g)$  is finite and  $a$  is prime to it.

## 2. PROOF: THE TAME CASE

Throughout this section, we assume that the characteristic  $p \neq 2, 3, 5$  and  $g$  is an automorphism of order 60 of a K3 surface.

By [7] Lemma 3.4 and 3.6,  $g$  cannot be of order 2.30, 3.20, 4.15 or 6.10. We first exclude the possibility 1.60.

**Lemma 2.1.**  $\text{ord}(g) \neq 1.60$ .

*Proof.* Suppose that  $\text{ord}(g) = 1.60$ . Then

$$[g^*] = [1, \zeta_{60} : 16, \eta_1, \dots, \eta_5]$$

where  $[\eta_1, \dots, \eta_5]$  is a combination of  $\zeta_{12} : 4, \zeta_{10} : 4, \zeta_5 : 4, \zeta_6 : 2, \zeta_4 : 2, \zeta_3 : 2, \pm 1$ , and the first eigenvalue corresponds to a  $g$ -invariant ample divisor.

Claim 1:  $[\eta_1, \dots, \eta_5] \neq [\zeta_{10} : 4, \pm 1], [\zeta_5 : 4, \pm 1]$ .

Suppose that  $[\eta_1, \dots, \eta_5] = [\zeta_{10} : 4, \pm 1]$  or  $[\zeta_5 : 4, \pm 1]$ . Then  $e(\text{Fix}(g^{30})) = \sum (-1)^j \text{Tr}(g^{30*} | H_{\text{et}}^j(X, \mathbb{Q}_l)) = -8$  and  $\text{Fix}(g^{30})$  consists of  $d$  smooth rational curves and a curve  $C_{d+5}$  of genus  $d+5$ . We have  $0 \leq d \leq 5$ , since each fixed curve gives an invariant vector in  $\dim H_{\text{et}}^2(X, \mathbb{Q}_l)$ . Note that  $e(\text{Fix}(g^2)) = \sum (-1)^j \text{Tr}(g^{2*} | H_{\text{et}}^j(X, \mathbb{Q}_l)) = 1$ . Since  $\text{Fix}(g^2) \subset \text{Fix}(g^{30})$ , we infer that  $\text{Fix}(g^2)$  consists of a point. Note that  $C_{d+5} \not\subset \text{Fix}(g^{10})$ , since  $e(\text{Fix}(g^{10})) = 16 > e(\text{Fix}(g^{30}))$ . If  $d = 1, 2$  or  $4$ , then  $g$  acts on the  $d$  smooth rational curves and  $g^2$  preserves at least one of them, hence fixes at least 2 points. If  $d = 3$ , then  $g$  must rotate the 3 smooth rational curves and  $g^{10}$  acts on the curve  $C_8$  with 16 fixed points, which is impossible. If  $d = 0$ , then  $g^{10}$  gives an order 3 automorphism of the curve  $C_5$  with 16 fixed points, impossible. If  $d = 5$ , then  $g$  must rotate the 5 smooth rational curves and  $g^5$  preserves each of them, hence  $e(\text{Fix}(g^5)) \geq 10$ . But  $\sum (-1)^j \text{Tr}(g^{5*} | H_{\text{et}}^j(X, \mathbb{Q}_l)) \leq 8$ , contradicting the Lefschetz fixed point formula.

Claim 2:  $[\eta_1, \dots, \eta_5] \neq [\zeta_6 : 2, \pm 1, \pm 1, \pm 1], [\zeta_3 : 2, \pm 1, \pm 1, \pm 1]$ .

Suppose that  $[\eta_1, \dots, \eta_5] = [\zeta_6 : 2, \pm 1, \pm 1, \pm 1]$  or  $[\zeta_3 : 2, \pm 1, \pm 1, \pm 1]$ . This case can be handled similarly. We see that  $e(\text{Fix}(g^{30})) = -8$  and  $\text{Fix}(g^{30})$  consists of  $d$  smooth rational curves and a curve  $C_{d+5}$  of genus  $d+5$ ,  $0 \leq d \leq 5$ . We also see that  $e(\text{Fix}(g^2)) = 3$  and  $\text{Fix}(g^2)$  consists of either 3 points or a point and a  $\mathbb{P}^1$ . Note that  $C_{d+5} \not\subset \text{Fix}(g^{10})$ , since  $e(\text{Fix}(g^{10})) = 13 > e(\text{Fix}(g^{30}))$ . If  $d = 0$  or  $1$ , then  $g^{10}$  gives an order 3 automorphism of the curve  $C_{d+5}$  with at least 11 fixed points, which is impossible. If  $d = 2$ , then

$g^2$  preserves 2 smooth rational curves, hence fixes at least 4 points. If  $d = 3$ , then  $g$  must rotate the 3 smooth rational curves and  $g^{10}$  acts on the curve  $C_8$  with 13 fixed points, impossible. If  $d = 4$ , then  $g^3$  preserves each of them, hence  $e(\text{Fix}(g^3)) \geq 8$  or  $e(\text{Fix}(g^3)) = 8 + e(C_9) = -8$ , which is possible only if  $[g^*] = [1, \zeta_{60} : 16, \zeta_3 : 2, 1, 1, 1]$ . Then  $e(\text{Fix}(g)) = 5 > e(\text{Fix}(g^2))$ , but  $\text{Fix}(g)$  and  $\text{Fix}(g^2)$  consist of isolated points and some  $\mathbb{P}^1$ 's. If  $d = 5$ , then  $g$  must rotate the 5 smooth rational curves and  $g^5$  preserves each of them, hence  $e(\text{Fix}(g^5)) \geq 10$ . But  $\sum (-1)^j \text{Tr}(g^{5*} | H_{\text{et}}^j(X, \mathbb{Q}_l)) \leq 7$ , contradicting the Lefschetz fixed point formula.

Claim 3:  $[\eta_1, \dots, \eta_5] \neq [(\zeta_6 : 2).2, \pm 1], [(\zeta_3 : 2).2, \pm 1], [\zeta_6 : 2, \zeta_3 : 2, \pm 1]$ . Suppose that  $[\eta_1, \dots, \eta_5] = [(\zeta_6 : 2).2, \pm 1], [(\zeta_3 : 2).2, \pm 1]$  or  $[\zeta_6 : 2, \zeta_3 : 2, \pm 1]$ . Note that  $e(\text{Fix}(g^{30})) = -8$  and  $\text{Fix}(g^{30})$  consists of  $d$  smooth rational curves and a curve  $C_{d+5}$  of genus  $d + 5$ ,  $0 \leq d \leq 5$ . We see that  $e(\text{Fix}(g^2)) = 0$ . Since  $\text{Fix}(g^2) \subseteq \text{Fix}(g^{30})$ ,  $\text{Fix}(g^2) = \emptyset$ , thus  $\text{Fix}(g) = \emptyset$  and  $[g^*] = [1, \zeta_{60} : 16, (\zeta_3 : 2).2, -1]$ . Note that  $C_{d+5} \not\subseteq \text{Fix}(g^{10})$ , since  $e(\text{Fix}(g^{10})) = 10 > e(\text{Fix}(g^{30}))$ . If  $d = 0$ , then  $g^{10}$  gives an order 3 automorphism of the curve  $C_5$  with 10 fixed points, which is impossible. If  $d = 1, 2$  or  $4$ , then  $g^2$  preserves at least one smooth rational curve, hence fixes at least 2 points. If  $d = 3$ , then  $g$  must rotate the 3 smooth rational curves, hence  $g^{15}$  acts freely on the curve  $C_8$ , since  $e(\text{Fix}(g^{15})) = 6$ . But no genus 8 curve admits a free involution. If  $d = 5$ , then  $g$  must rotate the 5 smooth rational curves and  $g^5$  preserves each of them, hence  $e(\text{Fix}(g^5)) \geq 10$ . But  $\sum (-1)^j \text{Tr}(g^{5*} | H_{\text{et}}^j(X, \mathbb{Q}_l)) = 0$ .

Claim 4:  $[\eta_1, \dots, \eta_5] \neq [\zeta_4 : 2, \zeta_6 : 2, \pm 1], [\zeta_4 : 2, \zeta_3 : 2, \pm 1]$ . Suppose that  $[\eta_1, \dots, \eta_5] = [\zeta_4 : 2, \zeta_6 : 2, \pm 1]$  or  $[\zeta_4 : 2, \zeta_3 : 2, \pm 1]$ . In this case,  $e(\text{Fix}(g^{30})) = -12$  and  $\text{Fix}(g^{30})$  consists of  $d$  smooth rational curves and a curve  $C_{d+7}$  of genus  $d + 7$ ,  $0 \leq d \leq 3$ . We see that

$$e(\text{Fix}(g^2)) = \sum (-1)^j \text{Tr}(g^{2*} | H_{\text{et}}^j(X, \mathbb{Q}_l)) = -1 > e(\text{Fix}(g^{30})),$$

hence  $C_{d+7} \not\subseteq \text{Fix}(g^2)$  and  $e(\text{Fix}(g^2)) \geq 0$ .

Claim 5:  $[\eta_1, \dots, \eta_5] \neq [(\zeta_4 : 2).2, \pm 1]$ . Suppose that  $[\eta_1, \dots, \eta_5] = [(\zeta_4 : 2).2, \pm 1]$ . In this case,  $e(\text{Fix}(g^{30})) = -16$  and  $\text{Fix}(g^{30})$  consists of  $d$  smooth rational curves and a curve  $C_{d+9}$  of genus  $d + 9$ ,  $0 \leq d \leq 1$ . We see that  $e(\text{Fix}(g^2)) = -2 > e(\text{Fix}(g^{30}))$ , hence  $C_{d+9} \not\subseteq \text{Fix}(g^2)$  and  $e(\text{Fix}(g^2)) \geq 0$ .

Claim 6:  $[\eta_1, \dots, \eta_5] \neq [\zeta_4 : 2, \pm 1, \pm 1, \pm 1]$ . Suppose that  $[\eta_1, \dots, \eta_5] = [\zeta_4 : 2, \pm 1, \pm 1, \pm 1]$ . In this case,  $e(\text{Fix}(g^{30})) = -12$  and  $\text{Fix}(g^{30})$  consists of  $d$  smooth rational curves and a curve  $C_{d+7}$  of genus  $d + 7$ ,  $0 \leq d \leq 3$ . We see that

$$e(\text{Fix}(g^2)) = \sum (-1)^j \text{Tr}(g^{2*} | H_{\text{et}}^j(X, \mathbb{Q}_l)) = 2 > e(\text{Fix}(g^{30})),$$

hence  $\text{Fix}(g^2)$  consists of either 2 points or a  $\mathbb{P}^1$ , since  $\text{Fix}(g^2) \subset \text{Fix}(g^{30})$ . Since  $\text{Fix}(g) \subset \text{Fix}(g^2)$ , we infer that

$$e(\text{Fix}(g)) = 2 \text{ or } 0.$$

By computing  $[g^{15*}]$  and  $[g^{10*}]$ , we see that

$$e(\text{Fix}(g)) = e(\text{Fix}(g^{15})) \text{ and } e(\text{Fix}(g^{10})) = 12.$$

If  $d = 0$ , then  $g^{10}$  gives an order 3 automorphism of the curve  $C_7$  with 12 fixed points, which is impossible. If  $d = 2$ , then  $g^2$  preserves both smooth rational curves, hence  $e(\text{Fix}(g^2)) \geq 4$ . If  $d = 3$ , then  $g^2$  cannot preserve each of the 3 smooth rational curves, hence  $g$  must rotate the 3 smooth rational curves, then  $g^{15}$  preserves each of the 3 smooth rational curves, hence  $e(\text{Fix}(g^{15})) \geq 6$ . If  $d = 1$ , then  $g^{15}$  acts freely on the curve  $C_8$ . But no genus 8 curve admits a free involution.

Claim 7:  $[\eta_1, \dots, \eta_5] \neq [\pm 1, \pm 1, \pm 1, \pm 1, \pm 1]$ .

Suppose that  $[\eta_1, \dots, \eta_5] = [\pm 1, \pm 1, \pm 1, \pm 1, \pm 1]$ . In this case,  $e(\text{Fix}(g^{30})) = -8$  and  $\text{Fix}(g^{30})$  consists of  $d$  smooth rational curves and a curve  $C_{d+5}$  of genus  $d + 5$ ,  $0 \leq d \leq 5$ . We also compute

$$e(\text{Fix}(g^2)) = 6, \quad e(\text{Fix}(g^{15})) = e(\text{Fix}(g)), \quad e(\text{Fix}(g^{10})) = 16.$$

Since  $e(\text{Fix}(g^2)) > e(\text{Fix}(g^{30}))$ , we see that  $C_{d+5} \not\subset \text{Fix}(g^2)$  and

$$e(\text{Fix}(g^{15})) = e(\text{Fix}(g)) \leq e(\text{Fix}(g^2)) = 6.$$

If  $d = 0, 1, 2$ , then  $g^{10}$  gives an order 3 automorphism of the curve  $C_{d+5}$  with  $16 - 2d$  fixed points, which is impossible. If  $d = 4, 5$ , then either  $g^2$  or  $g^{15}$  preserves each of the  $d$  smooth rational curves, hence  $e(\text{Fix}(g^2)) \geq 2d \geq 8$  or  $e(\text{Fix}(g^{15})) \geq 2d \geq 8$ . Assume  $d = 3$ . If  $g$  rotates the 3 smooth rational curves or fixes each of them, then  $g^{15}$  fixes each of them, hence acts freely on the curve  $C_8$ , a contradiction. If  $g$  fixes exactly one of the 3 smooth rational curves, then  $g^2$  fixes each of them, hence acts freely on the curve  $C_8$ , then  $g$  acts freely on the curve  $C_8$  and  $e(\text{Fix}(g)) = 2$ , then  $g^{15}$  has  $e(\text{Fix}(g^{15})) = 2$ , hence acts freely on the curve  $C_8$ .

Now by Claim 1–7,

$$[g^*] = [1, \zeta_{60} : 16, \zeta_{12} : 4, \pm 1].$$

Thus,  $e(\text{Fix}(g^{30})) = -16$  and  $\text{Fix}(g^{30})$  consists of  $d$  smooth rational curves and a curve  $C_{d+9}$  of genus  $d + 9$ ,  $0 \leq d \leq 1$ . Note that  $e(\text{Fix}(g^{10})) = 14$ . If  $d = 0$ , then  $g^{10}$  acts on the curve  $C_9$  with 14 fixed points, too many for an order 3 automorphism. Thus  $d = 1$ . Let

$$h := g^{30}.$$

By Lemma 1.5,  $h$  is non-symplectic and has a  $h$ -invariant elliptic fibration  $\psi : X \rightarrow \mathbb{P}^1$  with 12 cuspidal fibers, and  $X^h$  consists of a section  $R$  of  $\psi$  and a curve  $C_{10}$  of genus 10 which is a 3-section passing through each cusp with multiplicity 3. Furthermore, we know that  $X/\langle h \rangle \cong \mathbb{F}_4$  a rational

ruled surface (see the proof of Lemma 1.5). Since the automorphism of  $X/\langle h \rangle$  induced by  $g$  preserves the unique ruling, we see that  $g$  preserves the fibration  $\psi : X \rightarrow \mathbb{P}^1$ . Since  $e(\text{Fix}(g^{10})) = 14$  and  $\text{Fix}(g^{10})$  is contained in  $R \cup C_{10}$ , we infer that  $g^{10}$  acts trivially on the base of  $\psi$  and  $\text{Fix}(g^{10})$  consists of  $R$  and the 12 cusps of the cuspidal fibres. Note that  $e(\text{Fix}(g^2)) = 4$ . Thus  $g$  preserves 2 cuspidal fibres and makes the remaining 10 cuspidal fibres form a single orbit. It follows that  $\text{Fix}(g)$  consists of 4 points and the last eigenvalue in  $[g^*]$  must be 1. Consider the action of  $g^{20}$ . Since  $g^{10}$  acts trivially on the base of  $\psi$ , so does  $g^{20}$ . Note that  $\text{Fix}(g^{10}) \subset \text{Fix}(g^{20})$  and  $e(\text{Fix}(g^{20})) = -6$ . Thus we infer that  $\text{Fix}(g^{20})$  contains no isolated points and consists of  $R$  and a curve  $C_5$  of genus 5. Note that  $C_5$  is a 2-section of  $\psi$  and intersects transversally  $C_{10}$  at the 10 cusps. The quotient

$$X' := X/\langle g^{12} \rangle$$

is a singular surface with  $K_{X'}$  numerically trivial.

Claim 8:  $X' = X/\langle g^{12} \rangle$  has four singular points, one of type  $\frac{1}{5}(3, 3)$  and three of type  $\frac{1}{5}(2, 4)$ .

First note that  $e(\text{Fix}(g^{12})) = 4$  and  $\text{Fix}(g^{12}) = \text{Fix}(g)$  consists of 2 points of  $R$  and 2 points of  $C_{10} \cap C_5$ . Since  $g^{12*}\omega_X = \zeta_5\omega_X$ ,  $\zeta_5 \in k$ , there are two types of local action,  $\frac{1}{5}(3, 3)$  and  $\frac{1}{5}(2, 4)$ , for the order 5 automorphism  $g^{12}$ . Let  $a$  and  $b$  be the number of points respectively of the two types. Then  $a + b = 4$ . Let  $\varepsilon : Y \rightarrow X'$  be a minimal resolution. Then

$$K_Y = \varepsilon^*K_{X'} - \sum D_p$$

where  $D_p$  is an effective  $\mathbb{Q}$ -divisor supported on the exceptional set of the singular point  $p \in X'$ . Thus

$$K_Y^2 = \sum D_p^2 = - \sum K_Y D_p.$$

See, e.g., Lemma 3.6 [5] for the formulas of  $D_p$  and  $K_Y D_p$ , which are valid for tame quotient singular points in positive characteristic. We compute

$$K_Y^2 = 10 - \rho(Y) = 10 - \{\rho(X') + a + 2b\} = 4 - a - 2b.$$

Note that  $K_Y D_p = \frac{9}{5}$  if  $p$  is of type  $\frac{1}{5}(3, 3)$ , and  $K_Y D_p = \frac{2}{5}$  if  $p$  is of type  $\frac{1}{5}(2, 4)$ . Thus  $a = 1$  and  $b = 3$ . This proves the claim.

Now by Claim 8, we compute that

$$K_Y = -\frac{3A}{5} - \sum_{i=1}^3 \frac{A_{1i} + 2A_{2i}}{5}$$

where  $A$  and  $A_{ji}$  are exceptional curves with  $A^2 = -5$ ,  $A_{1i}^2 = -2$ ,  $A_{2i}^2 = -3$ ,  $A_{1i} \cdot A_{2i} = 1$ . If the 2 points of  $R$  are of type  $\frac{1}{5}(2, 4)$ , then the proper transform  $R'$  of the image of  $R$  in  $X'$  has intersection number with  $K_Y$ ,

$$K_Y \cdot R' = -\frac{1}{5} - \frac{1}{5}, -\frac{1}{5} - \frac{2}{5} \text{ or } -\frac{2}{5} - \frac{2}{5},$$



none is an integer. If the 2 points of  $C_{10} \cap C_5$  are of type  $\frac{1}{5}(2, 4)$ , then the proper transform  $C'_5$  of the image of  $C_5$  in  $X'$  has intersection number with  $K_Y$  which cannot be an integer, a contradiction.  $\square$

**Remark 2.2.** Machida and Oguiso [8] proved Lemma 2.1 in the complex case. Their proof depends on the holomorphic Lefschetz formula and the integral lattice theory, hence does not extend to the positive characteristic.

**Lemma 2.3.** *If  $\text{ord}(g) = 5.12$ , then*

$$[g^*] = [1, \zeta_{12} : 4, 1, \zeta_{60} : 16]$$

*where the first eigenvalue corresponds to a  $g$ -invariant ample divisor.*

*Proof.* Suppose that  $\text{ord}(g) = 5.12$ . Since  $g^{12}$  is symplectic of order 5,

$$[g^{12*}] = [1, 1.5, (\zeta_5).4]$$

where the first eigenvalue corresponds to an invariant ample divisor. By Proposition 1.1,  $\zeta_{12} \in [g^*]$ . Thus we infer that

$$[g^*] = [1, \zeta_{12} : 4, \pm 1, \eta_1, \dots, \eta_{16}]$$

where  $[\eta_1, \dots, \eta_{16}]$  is a combination of  $\zeta_5 : 4, \zeta_{10} : 4, \zeta_{15} : 8, \zeta_{20} : 8, \zeta_{30} : 8, \zeta_{60} : 16$  and the first eigenvalue corresponds to a  $g$ -invariant ample divisor.

Assume that  $[\eta_1, \dots, \eta_{16}]$  contains  $[\zeta_{15} : 8]$  or  $[\zeta_{30} : 8]$ . Then

$$[g^{2*}] = [1, (\zeta_6 : 2).2, 1, \zeta_{15} : 8, \tau_1, \dots, \tau_8]$$

where  $[\tau_1, \dots, \tau_8]$  is a combination of  $\zeta_5 : 4, \zeta_{10} : 4, \zeta_{15} : 8$ , hence  $\sum \tau_j \geq -2$  and

$$\text{Tr}(g^{2*}|H_{\text{et}}^2(X, \mathbb{Q}_l)) = 1 + 2 + 1 + 1 + \sum \tau_j \geq 5 - 2 = 3.$$

On the other hand, we know that for any positive integer  $a$  dividing 12,  $\text{Fix}(g^a) \subset \text{Fix}(g^{12})$  and

$$-2 \leq \text{Tr}(g^{a*}|H_{\text{et}}^2(X, \mathbb{Q}_l)) \leq \text{Tr}(g^{12*}|H_{\text{et}}^2(X, \mathbb{Q}_l)) = 2.$$

Assume that  $[\eta_1, \dots, \eta_{16}]$  contains  $[\zeta_{20} : 8]$ . In this case,

$$[g^{2*}] = [1, (\zeta_6 : 2).2, 1, (\zeta_{10} : 4).2, \tau_1, \dots, \tau_8].$$

Since  $\sum \tau_j \geq -2$ ,

$$\text{Tr}(g^{2*}|H_{\text{et}}^2(X, \mathbb{Q}_l)) = 1 + 2 + 1 + 2 + \sum \tau_j \geq 6 - 2 = 4,$$

contradicting the inequality  $\text{Tr}(g^{2*}|H_{\text{et}}^2(X, \mathbb{Q}_l)) \leq \text{Tr}(g^{12*}|H_{\text{et}}^2(X, \mathbb{Q}_l))$ .

Assume that  $[\eta_1, \dots, \eta_{16}]$  is a combination of  $\zeta_5 : 4, \zeta_{10} : 4$ . Then

$$[g^{6*}] = [1, -1.4, 1, (\zeta_5 : 4).4].$$

Thus

$$\text{Tr}(g^{6*}|H_{\text{et}}^2(X, \mathbb{Q}_l)) = 1 - 4 + 1 - 4 < -2,$$

contradicting  $\text{Tr}(g^{2*}|H_{\text{et}}^2(X, \mathbb{Q}_l)) \geq -2$ . We have shown that

$$[g^*] = [1, \zeta_{12} : 4, \pm 1, \zeta_{60} : 16].$$

Now as in the proof of Lemma 2.1, we see that

$$h := g^{30}$$

is non-symplectic and has a  $h$ -invariant elliptic fibration  $\psi : X \rightarrow \mathbb{P}^1$  with 12 cuspidal fibres, and  $X^h$  consists of a section  $R$  of  $\psi$  and a curve  $C_{10}$  of genus 10 which is a 3-section passing through each cusp with multiplicity 3, and  $g$  preserves the fibration  $\psi : X \rightarrow \mathbb{P}^1$ . Moreover,  $g^{10}$  acts trivially on the base of  $\psi$  and

$$\text{Fix}(g^{10}) = R \cup \{\text{the cusps of the 12 cuspidal fibres}\},$$

$g$  preserves 2 cuspidal fibres and makes the remaining 10 cuspidal fibres to form a single orbit. In particular,  $\text{Fix}(g)$  consists of 4 points, hence by Lefschetz the 6-th eigenvalue in  $[g^*]$  must be 1.  $\square$

*Proof of Theorem 0.1.* Let

$$y^2 + x^3 + A(t_0, t_1)x + B(t_0, t_1) = 0$$

be the Weierstrass equation of the  $g$ -invariant elliptic pencil, where  $A$  (resp.  $B$ ) is a binary form of degree 8 (resp. 12). By the previous result,  $g$  leaves invariant the section  $R$  and the action of  $g$  on the base of the fibration  $\psi : X \rightarrow \mathbb{P}^1$  is of order 10. After a linear change of the coordinates  $(t_0, t_1)$  we may assume that  $g$  acts on the base by

$$g : (t_0, t_1) \mapsto (t_0, \zeta_{10}t_1).$$

We know that  $g$  preserves two cuspidal fibres  $F_0, F_\infty$  and makes the remaining 10 cuspidal fibres to form one orbit. Thus the discriminant polynomial

$$\Delta = -4A^3 - 27B^2 = ct_0^2t_1^2(t_1^{10} - t_0^{10})^2$$

for some constant  $c \in k$ , as it must have two double roots (corresponding to the fibres  $F_0, F_\infty$ ) and one orbit of double roots. We know that the zeros of  $A$  correspond to either cuspidal fibres or nonsingular fibres with “complex multiplication” automorphism of order 6. Since this set is invariant with respect to the order 10 action of  $g$  on the base, we see that the only possibility is  $A = 0$ . Then we obtain

$$B = at_0t_1(t_1^{10} - t_0^{10})$$

for some constant  $a$ , and write the above Weierstrass equation in the form

$$y^2 + x^3 + at_0t_1(t_1^{10} - t_0^{10}) = 0.$$

A suitable linear change of variables makes  $a = 1$  without changing the action of  $g$  on the base. Thus  $X \cong X_{60}$  as an elliptic surface. We may assume that

$$g^*\left(\frac{dx \wedge dt}{y}\right) = \zeta_{12} \frac{dx \wedge dt}{y}.$$

Since  $g^{10}$  is of order 6 and acts trivially on the base,

$$g^{10}(x, y, t_0, t_1) = (\zeta_3x, \zeta_2y, t_0, t_1).$$

Note that

$$\text{Fix}(g) = \{\text{the two cusps of } F_0 \text{ and } F_\infty\} \cup (R \cap F_0) \cup (R \cap F_\infty).$$

Analyzing the local action of  $g$  at the fixed point  $(x, y, t_0, t_1) = (0, 0, 1, 0)$ , we infer that

$$g(x, y, t_0, t_1) = (\zeta_{30}x, \zeta_{20}y, t_0, \zeta_{10}t_1).$$

Here we first determine the linear terms, then see that the higher degree terms must vanish. We have proved Theorem 0.1 in the positive characteristic case.

### 3. PROOF: THE COMPLEX CASE

Throughout this section,  $X$  is a complex K3 surface.

A non-projective K3 surface cannot admit a non-symplectic automorphism of finite order (see [13], [9]), and its automorphisms of finite order are symplectic, hence of order  $\leq 8$ . Thus,  $X$  is projective.

The proofs of Lemma 2.1 and 2.3 go word for word, once we replace  $H_{\text{et}}^2(X, \mathbb{Q}_l)$  by  $H^2(X, \mathbb{Z})$ , and Proposition 1.3 by the usual topological Lefschetz fixed point formula. The part “Proof of Theorem 0.1” also goes word for word.

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